# Secondary characteristic classes of singular foliations (arxiv.org/abs/2106.10078)

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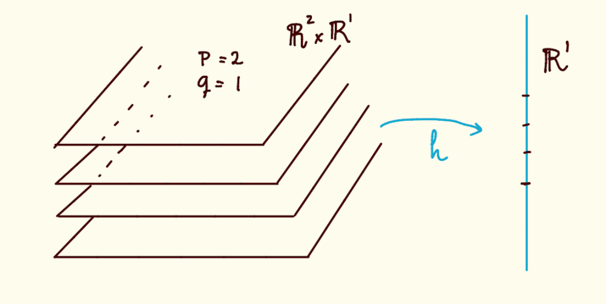
# Foliations

- The *standard codimension q foliation* of $\mathbb{R}^{p+q}$ is the decomposition of $\mathbb{R}^{p+q}$ into the level sets (*leaves*) of the map $\mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$.
- For $q = 1$ and $p + q = 3$:



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## Definition

On a manifold $M$, a (codimension $q$) foliation is a decomposition of $M$ as a union of submanifolds of codimension $q$ that locally looks like the standard foliation.
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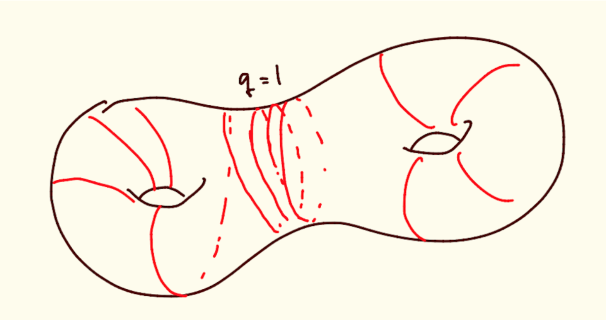

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The submanifolds are the *leaves* of the foliation.

- A *foliation map* $f\colon (M, \mathcal{F}) \to (N, \mathcal{F}')$ is a map that takes the leaves of $M$ to leaves of $N$.

# Smoothness

- Standing assumption: to make life easier, assume all objects and maps are smooth (that is, infinitely many derivatives) in this talk.
- There are interesting discussions about degree of regularity and existence of foliations, but they're not for today.

# Foliations as local level sets

- There are many alternative definitions for regular foliations. Haefliger provides one that is well suited to singular foliations.

## Definition

A *regular Haefliger foliation* (of codimension $q$) on $M$ is a collection of charts $\{h_\alpha \colon U_\alpha \to \mathbb{R}^q\}$ covering $M$, **each a submersion**, with level sets compatible on overlaps $U_\alpha \cap U_\beta$.

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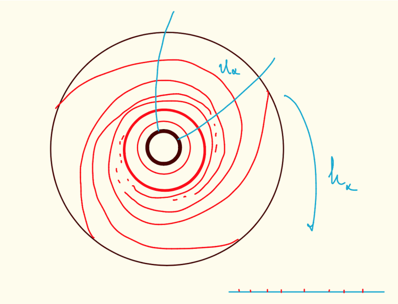

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- We can easily relax the definition by dropping the requirement that charts be submersions.

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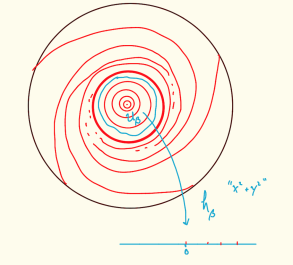
- The singular points are exactly those where the charts fail to be submersions.



# Why singular foliations?

- Foliations are natural objects, which show up all over, but they are often singular!
- For example, letting $n > q$, the level sets of a map $M^n \to N^q$ define a Haefliger foliation on $M$, of codimension $q$. The foliaton is regular if and only if the map is a submersion. (The local coordinate patches on $N$ pull back to Haefliger charts on $M$.)
- Consider the action of a (compact) Lie group $G$ on a space $M$. (For example, the action given by a representation $V$ of $G$.) The orbits will typically decompose $M$ into a foliation, and this foliation is typically not regular. (For example, at the origin in $V$ a representation of $G$.)
- You can't comb a hedgehog! (There are no regular codimension 1 foliations on $S^2$.)

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## The theory of foliations is global.

- The *local* theory of regular foliations is boring. Indeed, every point is geometrically the same! (Singular foliations have singularity type at singularities, obviously not boring, but...)
- The real interesting behavior is global, and here we have an interesting interplay with topology.
- Globally, we have questions like: How do the leaves wrap around each other? If you wander along a leaf, can you come back home? Or come close to home? (Ergodicity and so on.)
- For a famous example, the torii foliated by lines of either rational or irrational slope look very different globally, and this cannot be seen locally.

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# Detecting global non-triviality

- The theme of "locally boring, globally non-trivial" is a common one in maths. Here I would like to draw an analogy with the theory of vector bundles.
- A vector bundle $E$ is, by definition, locally trivial, but it can be difficult to decide whether $E$ is globally trivial.
- If you want to show that $E$ is non-trivial, one of the first things you should consider is whether it has non-vanishing *characteristic classes*.
- There is a theory of characteristic classes for foliations, even singular ones. Let me quickly describe it, while reviewing the same for vector bundles.

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## Characteristic classes

- A characteristic class is a 'natural' assignment from foliations on $M$ to elements of cohomology, $\mathcal{F} \mapsto c(\mathcal{F}) \in H^k(M)$ that respects foliation maps: For any map of foliations $f\colon (M, \mathcal{F}) \to (N, \mathcal{F}')$,

$$f^* c(\mathcal{F}') = c(\mathcal{F}).$$

  (Recall: a map of foliations is a map $f\colon M \to N$ that sends leaves to leaves.)
- Crucially for vector bundles, the characteristic classes are compatible with pullbacks: For any $f\colon M \to N$ and bundle $E$ over $N$, it holds that

$$c(f^* E) = f^* c(E) \in H^k(M).$$

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# Functoriality necessesitates singularities

- Given a regular foliation $(N, \mathcal{F}')$, and a map $f\colon M \to N$, the pullback foliation is frequently *not* regular.
  (Definition of pullback: if $(N, \mathcal{F}')$ is defined by charts $\{h_\alpha\colon U_\alpha \to \mathbb{R}^q\}$, then $(M, f^*\mathcal{F}')$ is defined by the pullback charts $\{h_\alpha \circ f\colon f^{-1}(U_\alpha) \to \mathbb{R}^q\}$.)
- For example, fix any foliation $(N, F')$ of codimension 1, and let $M = S^2$. There is no choice of codimension 1 foliation on $S^2$ (the can't-comb-a-hedgehog theorem...), so necessarily the pullback foliation of any map $S^2 \to N$ is singular.
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- There are classically two approaches to characteristic classes: you pull back forms from a classifying space, or you push forward curvatures of a connection.

# Characteristic classes via classifying spaces

## Theorem (Haefliger)

There exists a classifying space $B\Gamma_q$ for codimension-$q$ foliations. Foliations on $M$ 'up to homotopy' are in bijection with homotopy classes of maps $M \to B\Gamma_q$.

- As happens with vector bundles, the cohomology elements of $B\Gamma_q$ define characteristic classes, by pullback.
- This formulation is already compatible with singular foliations and all pullbacks. The functoriality of characteristic classes follows by definition.
- However, in terms of actually computing, this approach is difficult.

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# Characteristic classes via curvature

- If you actually want to compute characteristic classes, you're often well served to go the other direction: the Chern-Weil construction of characteristic classes.

# Characteristic classes via curvature

### Theorem (Bott, Haefliger)

Given a **regular** foliation $(M, \mathcal{F})$, choose connections 'adapted' to the foliation. One can construct explicit differential forms in $\Omega^*(M)$ that represent characteristic classes in cohomology. These are constructed by universal formula in the curvature and the connection forms.

- This map is very explicit, and furthermore, you now get precise form representatives for characteristic classes, and naturality *at the form level* (i.e. prior to descending to cohomology.)

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# Haefliger bundles

- There is another way to describe the Bott characteristic class construction. There is a natural assignment of principal bundles $\mathcal{H}(M, \mathcal{F})$, meaning that $\mathcal{H}(M, f^*\mathcal{F}') = f^*\mathcal{H}(N, \mathcal{F}')$ for any foliation map $(M, f^*\mathcal{F}') \to (N, \mathcal{F}')$.
- There is a tautological construction of 'characteristic classes' on $\mathcal{H}(M, \mathcal{F})$, we simply need a way to get them down to $M$.
  Any choice of metric $g$ on a regular foliation $(M, \mathcal{F})$ defines a (tautological) section $\lambda_g$ of the Haefliger bundle, and we can simply pull back by this section.
- The Haefliger bundle is well defined for singular foliations, what remains is the question of how to naturally construct sections, so as to pull back characteristic classes.

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# Adapted metrics

## Definition

Given a singular foliation $(M, \mathcal{F})$ whose set of regular points $\tilde{M}$ is dense in $M$, a metric on $\tilde{M}$ is *adapted* if the section $\lambda_g \colon \tilde{M} \to \mathcal{H}(\tilde{M}, \mathcal{F})$ extends smoothly to a section of $\mathcal{H}(M, \mathcal{F})$.

- If you have such a section, then you have succeeded in defining a form-level characteristic map for your singular foliation. You simply pull back by the extended section.
- Of course, this definition is only useful if adapted metrics exist! But they're not difficult to construct.

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# The existence of adapted metrics

## Theorem

*Let $(N, \mathcal{F}', g')$ be a (singular) foliation, with $g'$ an adapted metric. Then for any generically transverse immersion $f\colon M \to (N, F')$, the pullback metric $g = f^*g'$ on $\tilde{M}$ is adapted to the foliation $(M, f^*\mathcal{F}')$.*

- I haven't given you the definition of generically transverse immersion, but it's an easily satisfied condition that implies the regular set of $f^*\mathcal{F}'$ is dense.
  Subject to dimension constraints, any map $M \to N$ can be slightly perturbed to be a generically transverse immersion.
- Any metric on a regular foliaton is adapted. On the other hand, pullback foliations by generic transverse immersions frequently gain singularities. So the theorem already gives a large class of new foliations for our theory to apply to.

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Thanks for listening!